Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

(a) Write down the height of the ball above the ground at the instant it is hit by the bat. [1]

(b) Find the value of t when the ball hits the ground. [2]

(c) State an appropriate domain for t in this model. [2]

2. [Maximum mark: 5]

The ticket prices for a concert are shown in the following table.

Ticket Type	**Price (in Australian dollars, $)**
Adult	15
Child	10
Student	12

- A total of 600 tickets were sold.
- The total amount of money from ticket sales was $\$7816$.
- There were twice as many adult tickets sold as child tickets.

Let the number of adult tickets sold be x, the number of child tickets sold be y, and the number of student tickets sold be z.

(a) Write down three equations that express the information given above. [3]

(b) Find the number of each type of ticket sold. [2]

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3. [Maximum mark: 7]

Leo is investigating whether a six-sided die is fair. He rolls the die 60 times and records the observed frequencies in the following table:

Number on die	1	2	3	4	5	6
Observed frequency	8	7	6	15	12	12

Leo carries out a χ^2 goodness of fit test at a 5% significance level.

(a) Write down the null and alternative hypotheses. [1]

(b) Write down the degrees of freedom. [1]

(c) Write down the expected frequency of rolling a 1. [1]

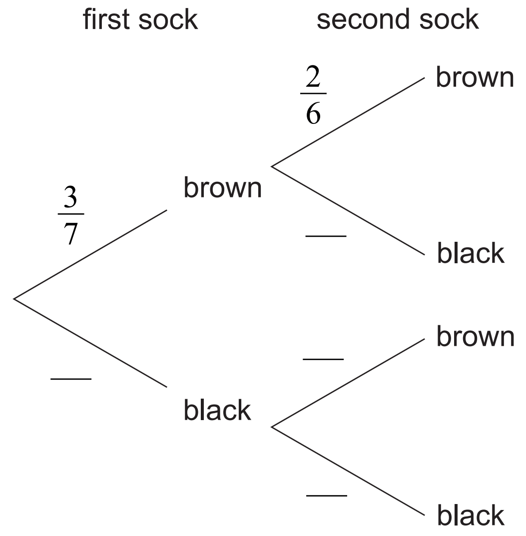
(d) Find the p-value for the test. [2]

(e) State the conclusion of the test. Give a reason for your answer. [2]

4. [Maximum mark: 7]

Karl has three brown socks and four black socks in his drawer. He takes two socks at random from the drawer.

(a) Complete the tree diagram. [2]



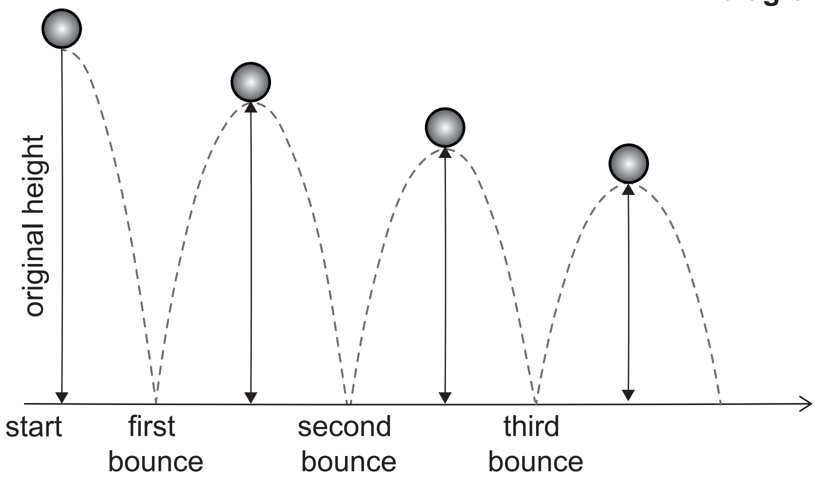
(b) Find the probability that Karl takes two socks of the same colour. [2]

(c) Given that Karl has two socks of the same colour find the probability that he has two brown socks. [3]

5. [Maximum mark: 7]

A ball is dropped from a height of 1.8 metres and bounces on the ground. The maximum height reached by the ball, after each bounce, is 85% of the previous maximum height.

diagram not to scale



(a) Show that the maximum height reached by the ball after it has bounced for the sixth time is $68\,\text{cm}$, to the nearest cm. [2]

(b) Find the number of times, after the first bounce, that the maximum height reached is greater than $10\,\text{cm}$. [2]

(c) Find the total **vertical** distance travelled by the ball from the point at which it is dropped until the fourth bounce. [3]

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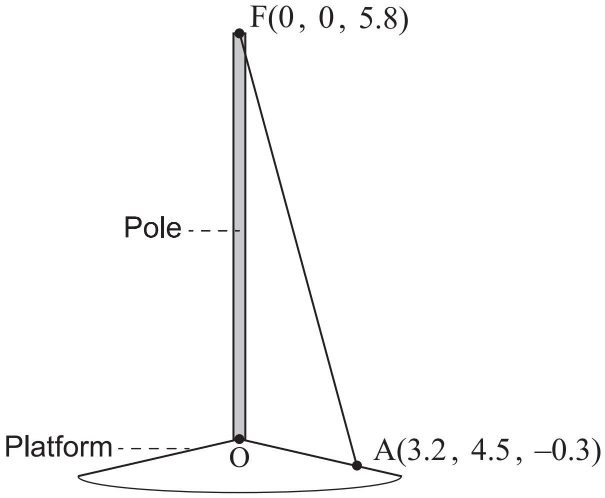
6. [Maximum mark: 8]

A vertical pole stands on a sloped platform. The bottom of the pole is used as the origin, O, of a coordinate system in which the top, F, of the pole has coordinates $(0, 0, 5.8)$. All units are in metres.

diagram not to scale



The pole is held in place by ropes attached at F.

One of these ropes is attached to the platform at point $A(3.2, 4.5, -0.3)$. The rope forms a straight line from A to F.

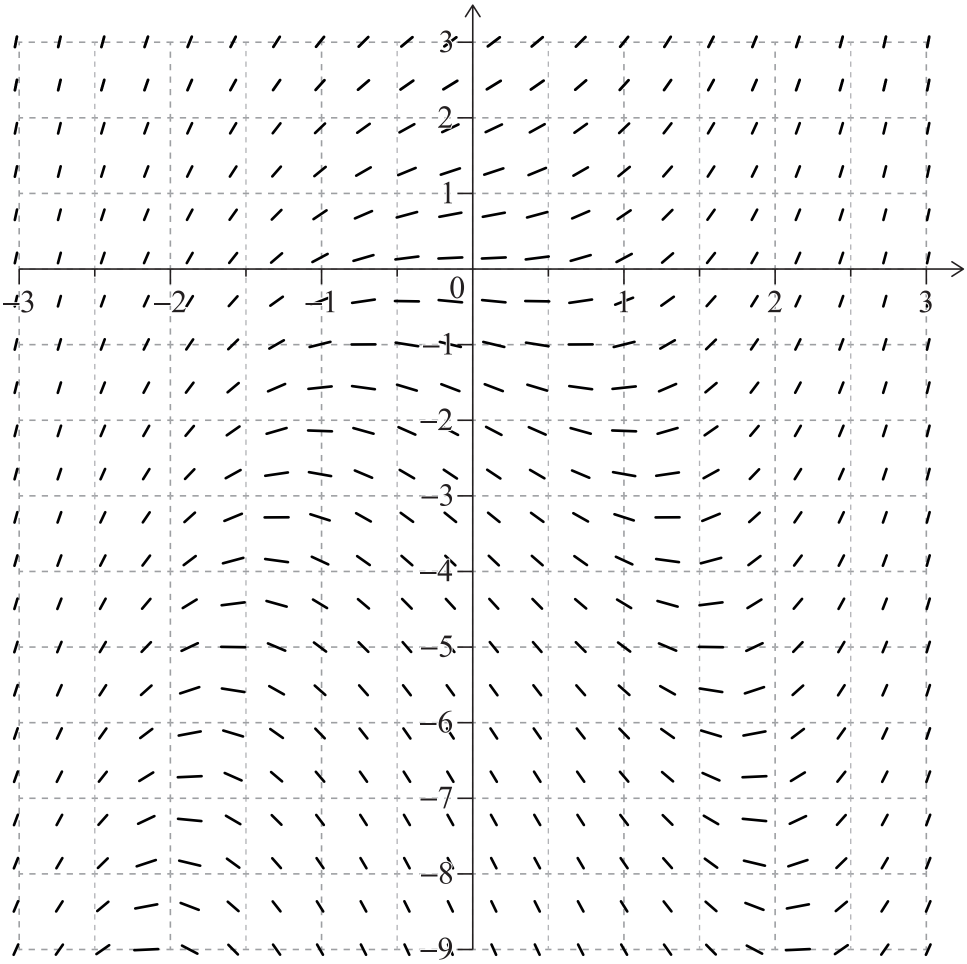
(a) Find $\overrightarrow{AF}$. [1]

(b) Find the length of the rope. [2]

(c) Find $F\hat{A}O$, the angle the rope makes with the platform. [5]

7. [Maximum mark: 4]

A slope field for the differential equation $\frac{dy}{dx} = x^2 + \frac{y}{2}$ is shown.



Some of the solutions to the differential equation have a local maximum point and a local minimum point.

(a) (i) Write down the equation of the curve on which all these maximum and minimum points lie.

(ii) Sketch this curve on the slope field. [2]

The solution to the differential equation that passes through the point $(0, -2)$ has both a local maximum point and a local minimum point.

(b) On the slope field, sketch the solution to the differential equation that passes through $(0, -2)$. [2]

(This question continues on the following page)

(Question 7 continued)

8. [Maximum mark: 6]

Consider the curve $y = 2x(4 - e^x)$.

(a) Find

(i) $\frac{dy}{dx}$.

(ii) $\frac{d^2y}{dx^2}$. [4]

The curve has a point of inflexion at (a, b).

(b) Find the value of a. [2]

9. [Maximum mark: 6]

A company produces bags of sugar with a labelled weight of $1\,\text{kg}$. The weights of the bags are normally distributed with a mean of $1\,\text{kg}$ and a standard deviation of $100\,\text{g}$. In an inspection, if the weight of a randomly chosen bag is less than $950\,\text{g}$ then the company fails the inspection.

(a) Find the probability that the company fails the inspection. [2]

A statistician in the company suggests it would be fairer if the company passes the inspection when the mean weight of five randomly chosen bags is greater than $950\,\text{g}$.

(b) Find the probability of passing the inspection if the statistician's suggestion is followed. [4]

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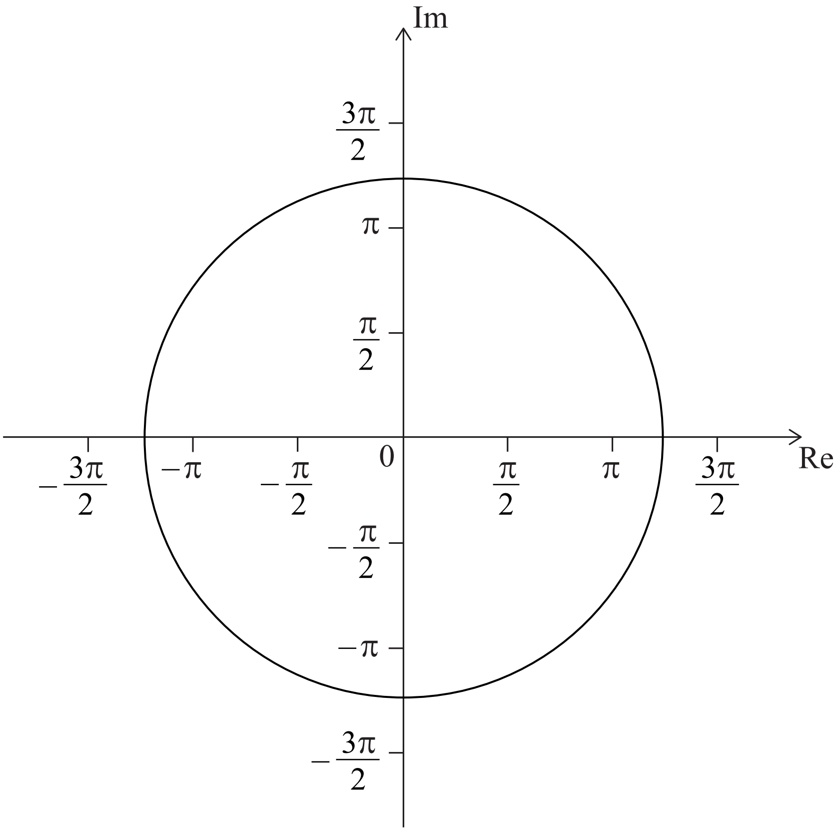
10. [Maximum mark: 7]

The following Argand diagram shows a circle centre 0 with a radius of 4 units.



A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$$z_\theta = \frac{1}{2}\theta e^{\theta i}, \text{ where } \theta \geq 0.$$

(a) Plot on the Argand diagram the points corresponding to

(i) $\theta = \frac{\pi}{2}$.

(ii) $\theta = \pi$.

(iii) $\theta = \frac{3\pi}{2}$. [3]

Consider the case where $|z_\theta| = 4$.

(b) (i) Find this value of θ.

(ii) For this value of θ, plot the approximate position of z_θ on the Argand diagram. [4]

(This question continues on the following page)

(Question 10 continued)

11. [Maximum mark: 5]

The matrix $\boldsymbol{M} = \begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}$ has eigenvalues -0.5 and 1.

(a) Find an eigenvector corresponding to the eigenvalue of 1. Give your answer in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, where $a,\ b \in \mathbb{Z}$. [3]

A switch has two states, A and B. Each second it either remains in the same state or moves according to the following rule: If it is in state A it will move to state B with a probability of 0.8 and if it is in state B it will move to state A with a probability of 0.7.

(b) Using your answer to (a), or otherwise, find the long-term probability of the switch being in state A. Give your answer in the form $\frac{c}{d}$, where $c,\ d \in \mathbb{Z}^{+}$. [2]

12. [Maximum mark: 8]

The strength of earthquakes is measured on the Richter magnitude scale, with values typically between 0 and 8 where 8 is the most severe.

The Gutenberg–Richter equation gives the average number of earthquakes per year, N, which have a magnitude of at least M. For a particular region the equation is

$$\log_{10} N = a - M \text{, for some } a \in \mathbb{R}.$$

This region has an average of 100 earthquakes per year with a magnitude of at least 3.

(a) Find the value of a. [2]

The equation for this region can also be written as $N = \frac{b}{10^M}$.

(b) Find the value of b. [2]

Within this region the most severe earthquake recorded had a magnitude of 7.2.

(c) Find the average number of earthquakes in a year with a magnitude of at least 7.2. [1]

The number of earthquakes in a given year with a magnitude of at least 7.2 can be modelled by a Poisson distribution, with mean N. The number of earthquakes in one year is independent of the number of earthquakes in any other year.

Let Y be the number of years between the earthquake of magnitude 7.2 and the next earthquake of at least this magnitude.

(d) Find $\mathrm{P}(Y > 100)$. [3]

13. [Maximum mark: 5]

At $1{:}00\ \text{pm}$ a ship is $1\,\text{km}$ east and $4\,\text{km}$ north of a harbour. A coordinate system is defined with the harbour at the origin. The position vector of the ship at $1{:}00\ \text{pm}$ is given by $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

The ship has a constant velocity of $\begin{pmatrix} 1.2 \\ -0.6 \end{pmatrix}$ kilometres per hour (km h^{-1}).

(a) Write down an expression for the position vector $\boldsymbol{r}$ of the ship, t hours after $1{:}00\ \text{pm}$. [1]

(b) Find the time at which the bearing of the ship from the harbour is 045°. [4]

14. [Maximum mark: 8]

(a) (i) Expand $\left(\frac{1}{u}+1\right)^2$.

(ii) Find $\int\left(\frac{1}{(x+2)}+1\right)^2 \mathrm{d}x$.

[4]

The region bounded by $y=\frac{1}{(x+2)}+1$, $x=0$, $x=2$ and the x-axis is rotated through 2π about the x-axis to form a solid.

(b) Find the volume of the solid formed. Give your answer in the form $\frac{\pi}{4}(a+b\ln(c))$, where a, b, $c\in\mathbb{Z}$.

[4]

15. [Maximum mark: 7]

The number of cars arriving at a junction in a particular town in any given minute between $9{:}00$ am and $10{:}00$ am is historically known to follow a Poisson distribution with a mean of 5.4 cars per minute.

A new road is built near the town. It is claimed that the new road has decreased the number of cars arriving at the junction.

To test the claim, the number of cars, X, arriving at the junction between $9{:}00$ am and $10{:}00$ am on a particular day will be recorded. The test will have the following hypotheses:

H_0: the mean number of cars arriving at the junction has not changed,
H_1: the mean number of cars arriving at the junction has decreased.

The alternative hypothesis will be accepted if $X \leq 300$.

(a) Assuming the null hypothesis to be true, state the distribution of X. [1]

(b) Find the probability of a Type I error. [2]

(c) Find the probability of a Type II error, if the number of cars now follows a Poisson distribution with a mean of 4.5 cars per minute. [4]

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16. [Maximum mark: 7]

The wind chill index W is a measure of the temperature, in $^{\circ}\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $(\mathrm{km\,h^{-1}})$ is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

(b) Find the rate of change of W at this time. [5]

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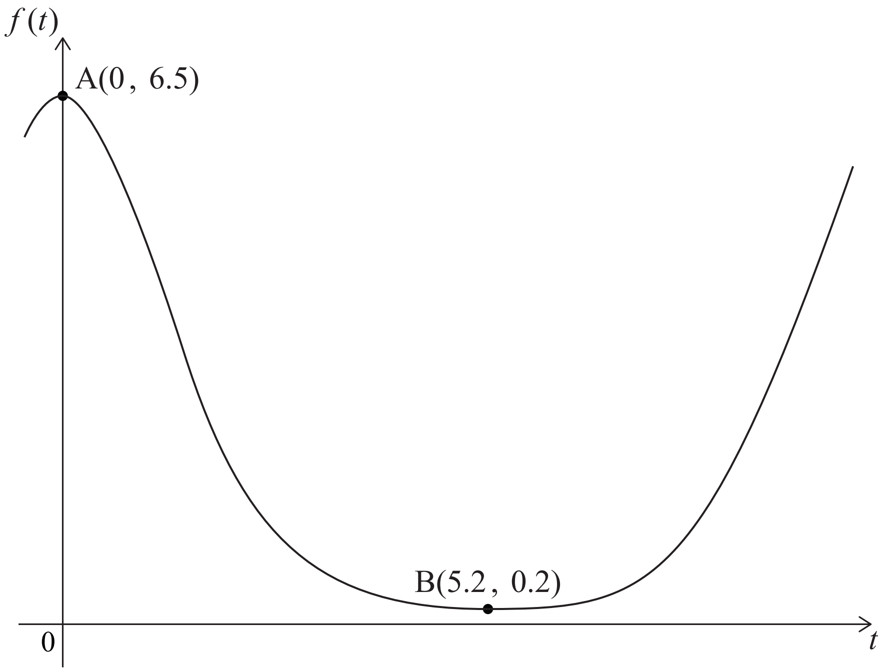
17. [Maximum mark: 8]

A function f is of the form $f(t) = p\mathrm{e}^{q\cos(rt)}$, $p, q, r \in \mathbb{R}^+$. Part of the graph of f is shown.



The points A and B have coordinates $\mathrm{A}(0, 6.5)$ and $\mathrm{B}(5.2, 0.2)$, and lie on f.

The point A is a local maximum and the point B is a local minimum.

Find the value of p, of q and of r.

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