

Mathematics: applications and interpretation
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 19]

Give your answers in parts (a), (d)(i), (e) and (f) to the nearest dollar.

Daisy invested 37 000 Australian dollars (AUD) in a fixed deposit account with an annual interest rate of 6.4% compounded **quarterly**.

(a) Calculate the value of Daisy's investment after 2 years. [3]

After m months, the amount of money in the fixed deposit account has appreciated to more than 50 000 AUD.

(b) Find the minimum value of m , where $m \in \mathbb{N}$. [4]

Daisy is saving to purchase a new apartment. The price of the apartment is 200 000 AUD.

Daisy makes an initial payment of 25% and takes out a loan to pay the rest.

(c) Write down the amount of the loan. [1]

The loan is for 10 years, compounded monthly, with equal monthly payments of 1700 AUD made by Daisy at the end of each month.

(d) For this loan, find

(i) the amount of interest paid by Daisy.

(ii) the annual interest rate of the loan. [5]

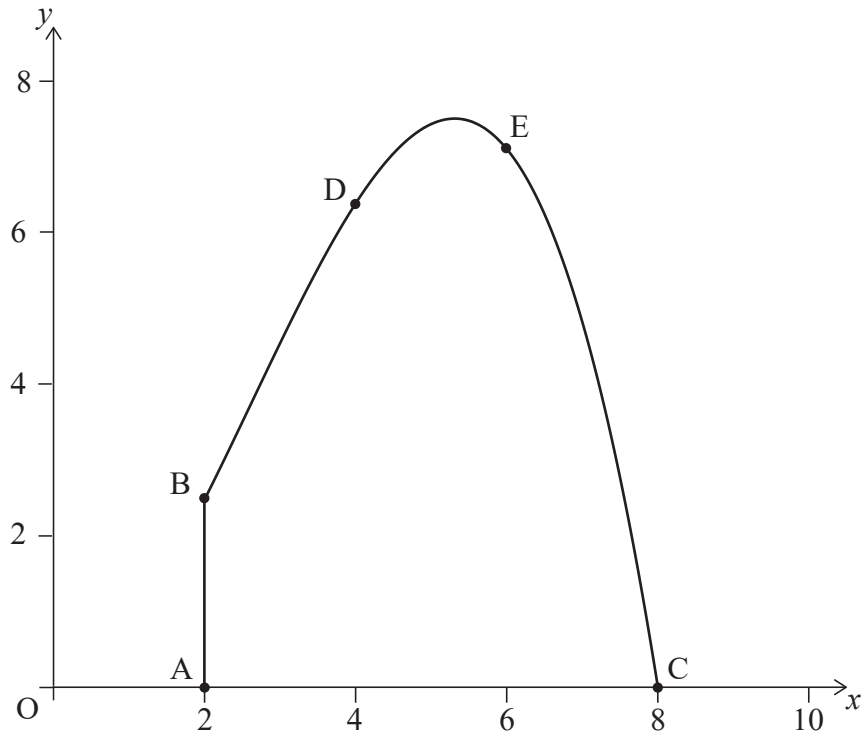
After 5 years of paying off this loan, Daisy decides to pay the **remainder** in one final payment.

(e) Find the amount of Daisy's final payment. [3]

(f) Find how much money Daisy saved by making one final payment after 5 years. [3]

2. [Maximum mark: 16]

The cross-sectional view of a tunnel is shown on the axes below. The line [AB] represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2$, $2 \leq x \leq 8$, relative to an origin O.



Point A has coordinates (2, 0), point B has coordinates (2, 2.4), and point C has coordinates (8, 0).

- (a) (i) Find $\frac{dy}{dx}$.
- (ii) Hence find the maximum height of the tunnel. [6]
- (b) Find the height of the tunnel when
- (i) $x = 4$.
- (ii) $x = 6$. [3]
- (c) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]
- (d) (i) Write down the integral which can be used to find the cross-sectional area of the tunnel.
- (ii) Hence find the cross-sectional area of the tunnel. [4]

3. [Maximum mark: 13]

The stopping distances for bicycles travelling at 20 km h^{-1} are assumed to follow a normal distribution with mean 6.76 m and standard deviation 0.12 m .

(a) Under this assumption, find, correct to four decimal places, the probability that a bicycle chosen at random travelling at 20 km h^{-1} manages to stop

(i) in less than 6.5 m .

(ii) in more than 7 m .

[3]

1000 randomly selected bicycles are tested and their stopping distances when travelling at 20 km h^{-1} are measured.

(b) Find, correct to four significant figures, the expected number of bicycles tested that stop between

(i) 6.5 m and 6.75 m .

(ii) 6.75 m and 7 m .

[3]

The measured stopping distances of the 1000 bicycles are given in the table.

Measured stopping distance	Number of bicycles
Less than 6.5 m	12
Between 6.5 m and 6.75 m	428
Between 6.75 m and 7 m	527
More than 7 m	33

It is decided to perform a χ^2 goodness of fit test at the 5% level of significance to decide whether the stopping distances of bicycles travelling at 20 km h^{-1} can be modelled by a normal distribution with mean 6.76 m and standard deviation 0.12 m .

(c) State the null and alternative hypotheses.

[2]

(d) Find the p -value for the test.

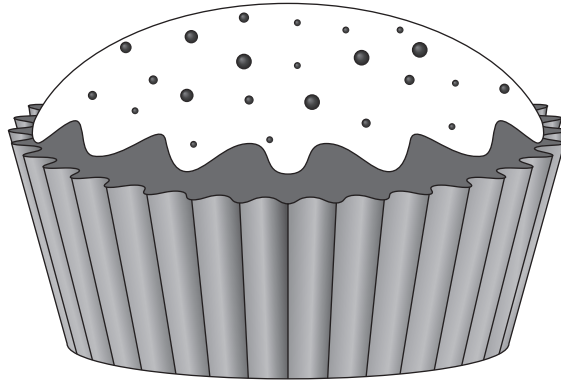
[3]

(e) State the conclusion of the test. Give a reason for your answer.

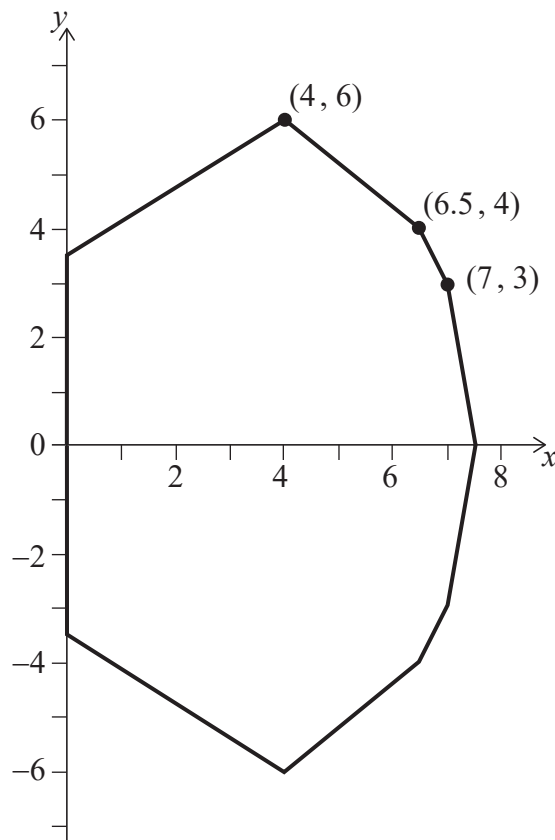
[2]

4. [Maximum mark: 14]

Charlotte decides to model the shape of a cupcake to calculate its volume.



From rotating a photograph of her cupcake she estimates that its cross-section passes through the points $(0, 3.5)$, $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$, where all units are in centimetres. The cross-section is symmetrical in the x -axis, as shown below:



She models the section from $(0, 3.5)$ to $(4, 6)$ as a straight line.

(a) Find the equation of the line passing through these two points.

[2]

(This question continues on the following page)

(Question 4 continued)

Charlotte models the section of the cupcake that passes through the points $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$ with a quadratic curve.

- (b) (i) Find the equation of the least squares regression quadratic curve for these four points.
- (ii) By considering the gradient of this curve when $x = 4$, explain why it may not be a good model. [3]

Charlotte thinks that a quadratic with a maximum point at $(4, 6)$ and that passes through the point $(7.5, 0)$ would be a better fit.

- (c) Find the equation of the new model. [4]

Believing this to be a better model for her cupcake, Charlotte finds the volume of revolution about the x -axis to estimate the volume of the cupcake.

- (d) (i) Write down an expression for her estimate of the volume as a sum of two integrals.
- (ii) Find the value of Charlotte's estimate. [5]

5. [Maximum mark: 13]

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix T is used to model this information, where $T = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

(b) Find the eigenvalues and eigenvectors of T . [5]

The matrix T can be written as a product of three matrices, PDP^{-1} , where D is a diagonal matrix.

(c) (i) Write down the matrix P .

(ii) Write down the matrix D . [2]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]

6. [Maximum mark: 18]

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a e^{bt} \cos t \\ a e^{bt} \sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O , where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t . [3]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$$a e^{bt} \sqrt{1 + b^2}. \quad [4]$$

At time $t = 0$, the displacement of the ice-skater is given by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$.

(c) Find the value of a and the value of b . [3]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

At a point P , the ice-skater is skating parallel to the y -axis for the first time.

(e) Find OP . [6]

7. [Maximum mark: 17]

A biologist introduces 100 rabbits to an island and records the size of their population (x) over a period of time. The population growth of the rabbits can be approximately modelled by the following differential equation, where t is time measured in years.

$$\frac{dx}{dt} = 2x$$

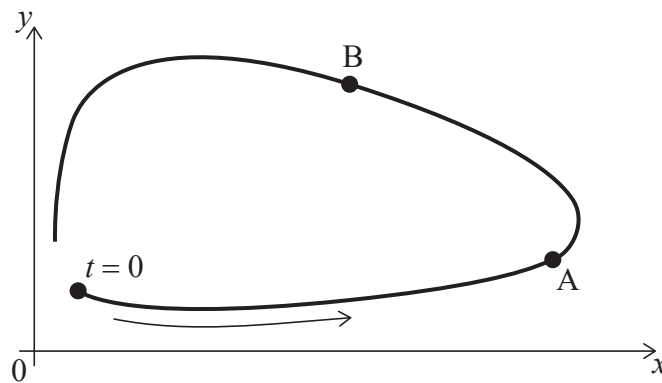
- (a) Find the population of rabbits 1 year after they were introduced. [5]

A population of 100 foxes is introduced to the island when the population of rabbits has reached 1000. The subsequent population growth of rabbits and foxes, where y is the population of foxes at time t , can be approximately modelled by the coupled equations:

$$\frac{dx}{dt} = x(2 - 0.01y)$$

$$\frac{dy}{dt} = y(0.0002x - 0.8)$$

- (b) Use Euler's method with a step size of 0.25, to find
- (i) the population of rabbits 1 year after the foxes were introduced.
 - (ii) the population of foxes 1 year after the foxes were introduced. [6]
- (c) The graph of the population sizes, according to this model, for the first 4 years after the foxes were introduced is shown below.



Describe the changes in the populations of rabbits and foxes for these 4 years at

- (i) point A.
 - (ii) point B. [3]
- (d) Find the non-zero equilibrium point for the populations of rabbits and foxes. [3]

References: