Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 9 November 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z) = \dfrac{e^z + e^{-z}}{2}$ and $g(z) = \dfrac{e^z - e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u, such that $t,\ u \in \mathbb{R}$.

(a) Verify that $u = f(t)$ satisfies the differential equation $\dfrac{d^2u}{dt^2} = u$. [2]

(b) Show that $(f(t))^2 + (g(t))^2 = f(2t)$. [3]

(c) Using $e^{iu} = \cos u + i\sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

(i) $f(iu)$; [3]

(ii) $g(iu)$. [2]

(d) Hence find, and simplify, an expression for $(f(iu))^2 + (g(iu))^2$. [2]

(e) Show that $(f(t))^2 - (g(t))^2 = (f(iu))^2 - (g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos\theta, \sin\theta)$ defines points on the unit circle with equation $x^2 + y^2 = 1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta), g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2 - y^2 = 1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2 - y^2 = 1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2 - y^2 = 1$ can be rotated to coincide with the curve defined by $xy = k,\ k \in \mathbb{R}$.

(g) Find the possible values of k. [5]

2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{dx}{dt} = x - y \text{ and } \frac{dy}{dt} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

(a) (i) By solving the differential equation $\frac{dy}{dt} = y$, show that $y = Ae^t$ where A is a constant. [3]

(ii) Show that $\frac{dx}{dt} - x = -Ae^t$. [1]

(iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a = -1$.

(b) (i) By differentiating $\frac{dy}{dt} = -x + y$ with respect to t, show that $\frac{d^2y}{dt^2} = 2\frac{dy}{dt}$. [3]

(ii) By substituting $Y = \frac{dy}{dt}$, show that $Y = Be^{2t}$ where B is a constant. [3]

(iii) Hence find y as a function of t. [2]

(iv) Hence show that $x = -\frac{B}{2}e^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

(c) (i) Show that $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = Fe^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

(ii) Find the two values for λ that satisfy $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2.

(iii) Verify that $y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References: