Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

Tuesday 2 November 2021 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

14 pages

8821 – 7102

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

In Lucy's music academy, eight students took their piano diploma examination and achieved scores out of 150. For her records, Lucy decided to record the average number of hours per week each student reported practising in the weeks prior to their examination. These results are summarized in the table below.

Average weekly practice time (h)	28	13	45	33	17	29	39	36
Diploma score (D)	115	82	120	116	79	101	110	121

(a) Find Pearson's product-moment correlation coefficient, r, for these data. [2]

(b) The relationship between the variables can be modelled by the regression equation $D = ah + b$. Write down the value of a and the value of b. [1]

(c) One of these eight students was disappointed with her result and wished she had practised more. Based on the given data, determine how her score could have been expected to alter had she practised an extra five hours per week. [2]

(d) Lucy asserts that the number of hours a student practises has a direct effect on their final diploma result. Comment on the validity of Lucy's assertion. [1]

Lucy suspected that each student had not been practising as much as they reported. In order to compensate for this, Lucy deducted a fixed number of hours per week from each of the students' recorded hours.

(e) State how, if at all, the value of r would be affected. [1]

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(Question 1 continued)

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Answers written on this page
will not be marked.

2. [Maximum mark: 5]

Consider a triangle ABC, where $\mathrm{AC} = 12$, $\mathrm{CB} = 7$ and $\mathrm{B\hat{A}C} = 25^{\circ}$.

Find the smallest possible perimeter of triangle ABC.

3. [Maximum mark: 7]

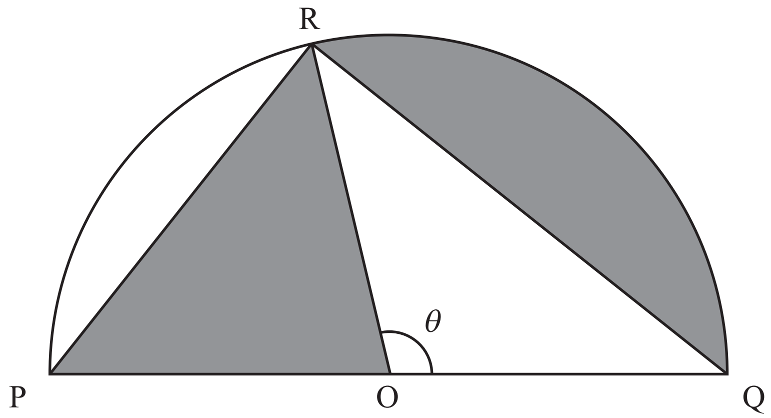
A factory manufactures lamps. It is known that the probability that a lamp is found to be defective is 0.05. A random sample of 30 lamps is tested.

(a) Find the probability that there is at least one defective lamp in the sample. [3]

(b) Given that there is at least one defective lamp in the sample, find the probability that there are at most two defective lamps. [4]

4. [Maximum mark: 6]

The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $\mathrm{PQ} = 2r$ and $\mathrm{R\hat{O}Q} = \theta$, where $0 < \theta < \pi$.



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of θ. [1]

5. [Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, u_1. [2]

(b) Find S_∞. [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

6. [Maximum mark: 8]

(a) Prove the identity $(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$. [2]

The equation $2x^2 - 5x + 1 = 0$ has two real roots, α and β.

Consider the equation $x^2 + mx + n = 0$, where $m, n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

(b) Without solving $2x^2 - 5x + 1 = 0$, determine the values of m and n. [6]

7. [Maximum mark: 6]

A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \arccos x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The median of this distribution is m.

(a) Determine the value of m. [2]

(b) Given that $P(|X - m| \le a) = 0.3$, determine the value of a. [4]

8. [Maximum mark: 8]

Consider the curve C given by $y = x - xy\ln(xy)$ where $x > 0$, $y > 0$.

(a) Show that $\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\left(1 + \ln(xy)\right) = 1$. [3]

(b) Hence find the equation of the tangent to C at the point where $x = 1$. [5]

Do **not** write solutions on this page.

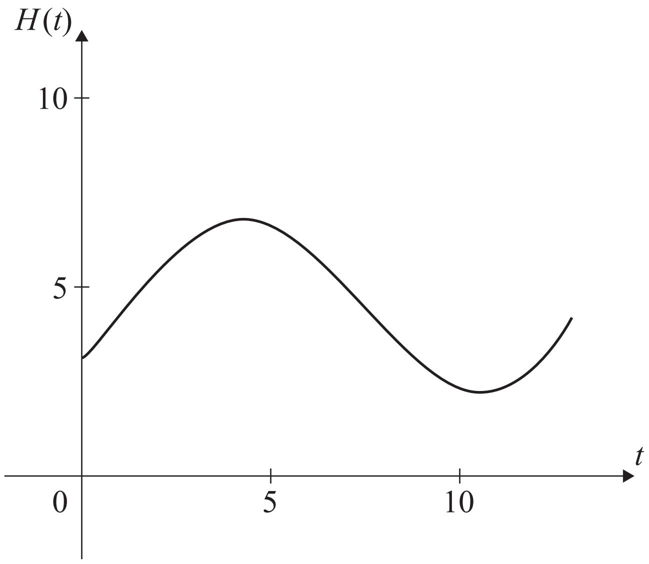
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 15]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a\sin(b(t-c)) + d$, where t is the number of hours after midnight, and a, b, c and d are constants, where $a > 0$, $b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.



The first high tide occurs at 04:30 and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between $2.2\,\text{m}$ and $6.8\,\text{m}$.

All heights are given correct to one decimal place.

(a) Show that $b = \frac{\pi}{6}$. [1]

(b) Find the value of a. [2]

(c) Find the value of d. [2]

(d) Find the smallest possible value of c. [3]

(e) Find the height of the water at 12:00. [2]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than 5 metres. [3]

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(Question 9 continued)

A fisherman notes that the water height at nearby Folkestone harbour follows the same sinusoidal pattern as that of Dungeness harbour, with the exception that high tides (and low tides) occur 50 minutes earlier than at Dungeness.

(g) Find a suitable equation that may be used to model the tidal height of water at Folkestone harbour. [2]

10. [Maximum mark: 18]

Consider the function $f(x)=\dfrac{x^2-x-12}{2x-15}$, $x\in\mathbb{R}$, $x\neq\dfrac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y=ax+b$ where a, $b\in\mathbb{Q}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30\leq x\leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

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11. [Maximum mark: 21]

Three points $\mathrm{A}(3, 0, 0)$, $\mathrm{B}(0, -2, 0)$ and $\mathrm{C}(1, 1, -7)$ lie on the plane Π_1.

(a) (i) Find the vector $\overrightarrow{\mathrm{AB}}$ and the vector $\overrightarrow{\mathrm{AC}}$.

(ii) Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a, b, c, d \in \mathbb{Z}$. [7]

Plane Π_2 has equation $3x - y + 2z = 2$.

(b) The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. [2]

(c) The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

(i) Show that at the point P, $\lambda = \frac{3}{4}$.

(ii) Hence find the coordinates of P. [3]

(d) The point $\mathrm{B}(0, -2, 0)$ lies on L.

(i) Find the reflection of the point B in the plane Π_3.

(ii) Hence find the vector equation of the line formed when L is reflected in the plane Π_3. [9]

References:

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