Mathematics: analysis and approaches
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages
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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ **for various values of** a **and** b, **where** $a, b \in \mathbb{N}$.

(a) On the same set of axes, sketch the following curves for $-2 \le x \le 2$ and $-2 \le y \le 2$, clearly indicating any points of intersection with the coordinate axes.

(i) $y^2 = x^3$, $x \ge 0$ [2]

(ii) $y^2 = x^3 + 1$, $x \ge -1$ [2]

(b) (i) Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. [1]

(ii) By considering each curve from part (a), identify two key features that would distinguish one curve from the other. [1]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \ge -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

(c) By varying the value of b, suggest two key features common to these curves. [2]

Next, consider the curve $y^2 = x^3 + x$, $x \ge 0$.

(d) (i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \pm\dfrac{3x^2+1}{2\sqrt{x^3+x}}$, for $x > 0$. [3]

(ii) Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. [1]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x-coordinate.

(e) Find the value of this x-coordinate, giving your answer in the form $x = \sqrt{\dfrac{p\sqrt{3}+q}{r}}$, where p, q, $r \in \mathbb{Z}$. [7]

(This question continues on the following page)

(Question 1 continued)

$P(x, y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q.

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $P(-1, -1)$ lies on C.

(f) (i) Find the equation of the tangent to C at P. [2]

(ii) Hence, find the coordinates of the rational point Q where this tangent intersects C, expressing each coordinate as a fraction. [2]

(g) The point $S(-1, 1)$ also lies on C. The line [QS] intersects C at a further point. Determine the coordinates of this point. [5]

2. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q, $r \in \mathbb{R}$, has roots α, β and γ.

(a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma.$$ [3]

(b) (i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

(c) Given that $p^2 < 3q$, deduce that α, β and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then α, β and γ are all real.

(e) (i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

(ii) Explain why the equation will have at least one real root for all values of q. [1]

(This question continues on the following page)

(Question 2 continued)

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where p, q, r, $s \in \mathbb{R}$, has roots α, β, γ and δ.

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

(f) (i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. [3]

(ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]

(g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

(h) (i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]

(ii) Write down the integer root of this equation. [1]

(iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

References: