Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

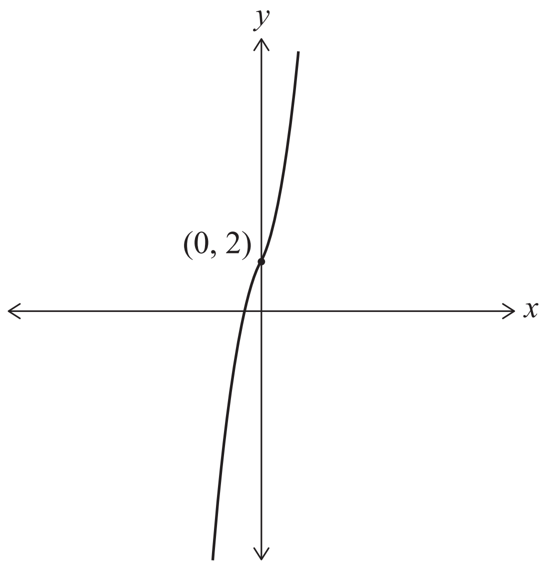
1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

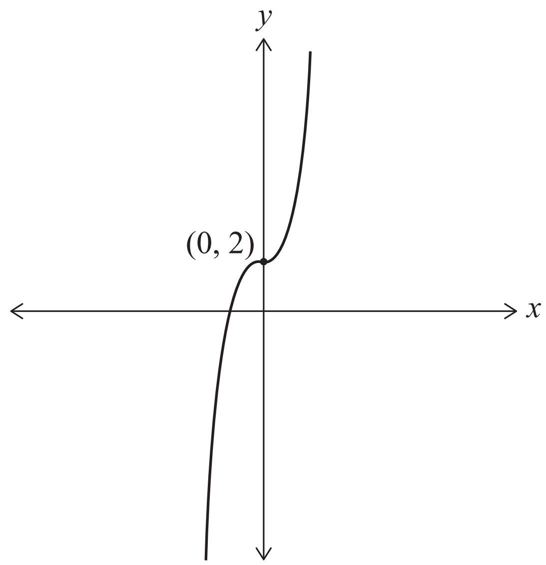
Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.

$c = -1$



$c = 0$



(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y-intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

(c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has

(i) a point of inflexion with zero gradient; [1]

(ii) one local maximum point and one local minimum point; [2]

(iii) no points where the gradient is equal to zero. [1]

(d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that

(i) the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]

(ii) the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]

(e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has

(i) exactly one x-axis intercept; [2]

(ii) exactly two x-axis intercepts; [2]

(iii) exactly three x-axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c,\ d \in \mathbb{R}$.

(f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x-axis intercept, explaining your reasoning. [6]

2. [Maximum mark: 28]

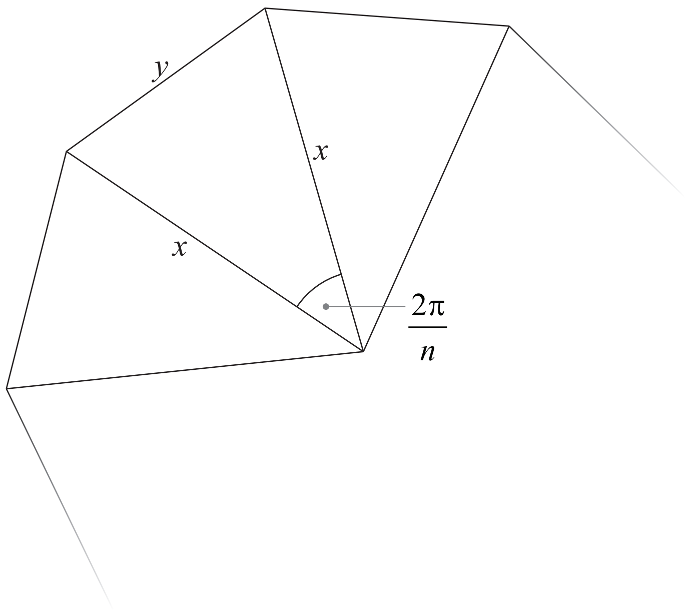
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P.

(a) Find the side length, s, where $s > 0$, of a square such that $A = P$. [3]

An n-sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n-sided regular polygon is shown in the following diagram.



(b) Write down, in terms of x and n, an expression for the area, A_T, of one of these isosceles triangles. [1]

(c) Show that $y = 2x\sin\frac{\pi}{n}$. [2]

Consider a n-sided regular polygon such that $A = P$.

(d) Use the results from parts (b) and (c) to show that $A = P = 4n\tan\frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a, b and $\sqrt{a^2+b^2}$, where $a \ge b$, such that $A = P$.

(f) Show that $a=\frac{8}{b-4}+4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which a, b, A, $P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

References: